

# SCALAR CURVATURE RIGIDITY WITH A VOLUME CONSTRAINT

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ABSTRACT. Motivated by Brendle-Marques-Neves' counterexample to the Min-Oo's conjecture, we prove a volume constrained scalar curvature rigidity theorem which applies to the hemisphere.

## 1. INTRODUCTION

Recently, Brendle, Marques and Neves [6] have solved the long-standing Min-Oo's conjecture [15] by constructing a counterexample.

**Theorem 1.1** (Brendle, Marques and Neves [6]). *Suppose  $n \geq 3$ . Let  $\bar{g}$  be the standard metric on the hemisphere  $\mathbb{S}_+^n$ . There exists a smooth metric  $g$  on  $\mathbb{S}_+^n$ , which can be made to be arbitrarily close to  $\bar{g}$  in the  $C^\infty$ -topology, satisfying*

- *the scalar curvature of  $g$  is at least that of  $\bar{g}$  at each point in  $\mathbb{S}_+^n$*
- *$g$  and  $\bar{g}$  agree in a neighborhood of  $\partial\mathbb{S}_+^n$ ,*

*but  $g$  is not isometric to  $\bar{g}$ .*

In this paper, we observe that if the metric  $g$  in Theorem 1.1 is assumed to satisfy an additional volume constraint, then it must be isometric to  $\bar{g}$ . Precisely, we have

**Theorem 1.2.** *Let  $\bar{g}$  be the standard metric on  $\mathbb{S}_+^n$ . Let  $g$  be another metric on  $\mathbb{S}_+^n$  with the properties*

- *$R(g) \geq R(\bar{g})$  in  $\mathbb{S}_+^n$*
- *$H(g) \geq H(\bar{g})$  on  $\partial\mathbb{S}_+^n$*
- *$g$  and  $\bar{g}$  induce the same metric on  $\partial\mathbb{S}_+^n$*

*where  $R(g)$ ,  $R(\bar{g})$  are the scalar curvature of  $g$ ,  $\bar{g}$ , and  $H(g)$ ,  $H(\bar{g})$  are the mean curvature of  $\Sigma$  in  $(\Omega, g)$ ,  $(\Omega, \bar{g})$ . Suppose in addition*

$$V(g) \geq V(\bar{g}),$$

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where  $V(g)$ ,  $V(\bar{g})$  are the volume of  $g$ ,  $\bar{g}$ . If  $\|g - \bar{g}\|_{C^2(\bar{\Omega})}$  is sufficiently small, then there is a diffeomorphism  $\varphi : \Omega \rightarrow \Omega$  with  $\varphi|_{\Sigma} = \text{id}$ , the identify map on  $\Sigma$ , such that  $\varphi^*(g) = \bar{g}$ .

Theorem 1.2 is indeed a special case of a more general result:

**Theorem 1.3.** *Let  $(\Omega, \bar{g})$  be an  $n$ -dimensional compact Riemannian manifold, of constant sectional curvature 1, with smooth boundary  $\Sigma$ . Suppose  $\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$  (i.e.  $\bar{\mathbb{I}} + \bar{H}\bar{\gamma}$  is positive semi-definite), where  $\bar{\gamma}$  is the induced metric on  $\Sigma$  and  $\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$  in  $(\Omega, \bar{g})$ . Suppose the first nonzero Neumann eigenvalue  $\mu$  of  $(\Omega, \bar{g})$  satisfies  $\mu > n - \frac{2}{n+1}$ .*

*Consider a nearby metric  $g$  on  $\Omega$  with the properties*

- $R(g) \geq n(n-1)$  where  $R(g)$  is the scalar curvature of  $g$
- $H(g) \geq \bar{H}$  where  $H(g)$  is the mean curvature of  $\Sigma$  in  $(\Omega, g)$
- $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$
- $V(g) \geq V(\bar{g})$  where  $V(g)$ ,  $V(\bar{g})$  are the volumes of  $g$ ,  $\bar{g}$ .

*If  $\|g - \bar{g}\|_{C^2(\bar{\Omega})}$  is sufficiently small, then there is a diffeomorphism  $\varphi$  on  $\Omega$  with  $\varphi|_{\Sigma} = \text{id}$ , such that  $\varphi^*(g) = \bar{g}$ .*

As a by-product of the method used to derive Theorem 1.3, we obtain a volume estimate for metrics close to the Euclidean metric in terms of the scalar curvature.

**Theorem 1.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\Sigma$ . Suppose  $\bar{\mathbb{I}} + \bar{H}\bar{\gamma} > 0$  (i.e.  $\bar{\mathbb{I}} + \bar{H}\bar{\gamma}$  is positive definite), where  $\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$  in  $\mathbb{R}^n$  and  $\bar{\gamma}$  is the metric on  $\Sigma$  induced from the Euclidean metric  $\bar{g}$ . Let  $g$  be another metric on  $\bar{\Omega}$  satisfying*

- $H(g) \geq \bar{H}$ , where  $H(g)$  is the mean curvature of  $\Sigma$  in  $(\Omega, g)$
- $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$ .

*Given any point  $a \in \mathbb{R}^n$ , there exists a constant  $\Lambda > \frac{\max_{q \in \bar{\Omega}} |q-a|^2}{4(n-1)}$ , depending only on  $\Omega$  and  $a$ , such that if  $\|g - \bar{g}\|_{C^3(\bar{\Omega})}$  is sufficiently small, then*

$$(1.1) \quad V(g) - V(\bar{g}) \geq \int_{\Omega} R(g) \Phi \, d\text{vol}_{\bar{g}}$$

*where  $\Phi(x) = -\frac{1}{4(n-1)}|x - a|^2 + \Lambda > 0$  on  $\bar{\Omega}$ .*

Theorem 1.4 may be compared to a previous theorem of Bartnik [2], which estimates the total mass [1] of an asymptotically flat metric that is a perturbation of the Euclidean metric.

**Theorem 1.5** (Bartnik [2]). *Let  $g$  be an asymptotically flat metric on  $\mathbb{R}^3$ . If  $g$  is sufficiently close to the Euclidean metric  $\bar{g}$  (in certain weighted Sobolev space), then*

$$(1.2) \quad 16\pi \mathbf{m}(g) \geq \int_{\mathbb{R}^3} R(g) \, d\text{vol}_g$$

where  $\mathbf{m}(g)$  is the total mass of  $g$ .

Our proofs of Theorems 1.2 - 1.4 follow a recent perturbation analysis of Brendle and Marques in [5], where they established a scalar curvature rigidity theorem for “small” geodesic balls in  $\mathbb{S}^n$ .

**Theorem 1.6** (Brendle and Marques [5]). *Let  $\Omega \subset \mathbb{S}^n$  be a geodesic ball of radius  $\delta$ . Suppose*

$$(1.3) \quad \cos \delta \geq \frac{2}{\sqrt{n+3}}.$$

Let  $\bar{g}$  be the standard metric on  $\mathbb{S}^n$ . Let  $g$  be another metric on  $\Omega$  with the properties

- $R(g) \geq n(n-1)$  at each point in  $\Omega$
- $H(g) \geq \bar{H}$  at each point on  $\partial\Omega$
- $g$  and  $\bar{g}$  induce the same metric on  $\partial\Omega$

where  $R(g)$  is the scalar curvature of  $g$ , and  $H(g), \bar{H}$  are the mean curvature of  $\partial\Omega$  in  $(\Omega, g), (\Omega, \bar{g})$ . If  $g - \bar{g}$  is sufficiently small in the  $C^2$ -norm, then  $\varphi^*(g) = \bar{g}$  for some diffeomorphism  $\varphi : \Omega \rightarrow \Omega$  such that  $\varphi|_{\partial\Omega} = \text{id}$ .

In Theorem 1.6, the condition (1.3) is equivalently to

$$(1.4) \quad \bar{H} \geq 4 \tan \delta$$

because the mean curvature  $\bar{H}$  of  $\partial B(\delta)$  is  $(n-1) \frac{\cos \delta}{\sin \delta}$ . As another application of the formulas in Section 2, we obtain a generalization of Theorem 1.6 to convex domains in  $\mathbb{S}^n$ .

**Theorem 1.7.** *Let  $\Omega \subset \mathbb{S}^n$  be a smooth domain contained in a geodesic ball  $B$  of radius less than  $\frac{\pi}{2}$ . Let  $\bar{g}$  be the standard metric on  $\mathbb{S}^n$ . Let  $\bar{\text{III}}, \bar{H}$  be the second fundamental form, the mean curvature of  $\partial\Omega$  in  $(\Omega, \bar{g})$ . Suppose  $\Omega$  is convex, i.e.  $\bar{\text{III}} \geq 0$ . At  $\partial\Omega$ , suppose*

$$(1.5) \quad \bar{H} \geq 4 \tan r$$

where  $r$  is the  $\bar{g}$ -distance to the center of  $B$ . Then the conclusion of Theorem 1.6 holds on  $\Omega$ .

Theorem 1.7 is an immediate consequence of Theorem 5.1 in Section 5. In a simpler setting, where the background metric  $\bar{g}$  is a flat metric, we have

**Theorem 1.8.** *Let  $\Omega$  be a compact manifold with smooth boundary  $\Sigma$ . Suppose there is a flat metric  $\bar{g}$  on  $\Omega$  such that  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$  (i.e.  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma}$  is positive semi-definite), where  $\bar{\mathbb{I}}\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$ , and  $\bar{\gamma}$  is the induced metric on  $\Sigma$ . Given another metric  $g$  on  $\Omega$  such that*

- $R(g) \geq 0$  on  $\Omega$
- $H(g) \geq \bar{H}$  at  $\Sigma$
- $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$ ,

*if  $\|g - \bar{g}\|_{C^2(\bar{\Omega})}$  is sufficiently small, then  $\varphi^*(g) = \bar{g}$  for some diffeomorphism  $\varphi : \Omega \rightarrow \Omega$  with  $\varphi|_{\Sigma} = \text{id}$ .*

Similar calculation at the infinitesimal level provides examples of compact 3-manifolds of nonnegative scalar curvature whose boundary surface does not have positive Gaussian curvature but still has positive Brown-York mass [7, 8]. We include this in the end of the paper to compare with known results in [17].

**Theorem 1.9.** *Let  $\Sigma \subset \mathbb{R}^n$  be a connected, closed hypersurface satisfying  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$ , where  $\bar{\mathbb{I}}\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$ , and  $\bar{\gamma}$  is the induced metric on  $\Sigma$ . Let  $\Omega$  be the domain enclosed by  $\Sigma$  in  $\mathbb{R}^n$ . Let  $h$  be any nontrivial  $(0, 2)$  symmetric tensor on  $\Omega$  satisfying*

$$(1.6) \quad \text{div}_{\bar{g}} h = 0, \quad \text{tr}_{\bar{g}} h = 0, \quad h|_{T\Sigma} = 0.$$

*Let  $\{g(t)\}_{|t| < \epsilon}$  be a 1-parameter family of metrics on  $\Omega$  satisfying*

$$(1.7) \quad g(0) = \bar{g}, \quad g'(0) = h, \quad R(g(t)) \geq 0, \quad g(t)|_{T\Sigma} = \bar{g}|_{T\Sigma}.$$

*Then*

$$(1.8) \quad \int_{\Sigma} \bar{H} d\sigma_{\bar{g}} > \int_{\Sigma} H(g(t)) d\sigma_{\bar{g}}$$

*for small  $t \neq 0$ , where  $H(g(t))$  is the mean curvature of  $\Sigma$  in  $(\Omega, g(t))$ .*

This paper is organized as follows. In Section 2, we derive a basic formula concerning a perturbed metric (Theorem 2.1), which corresponds to [5, Theorem 10] of Brendle and Marques. In Section 3, we prove Theorem 1.3, which implies Theorem 1.2. In Section 4, we give a proof of Theorem 1.4. In Section 5, we consider other applications of the formulas in Section 2 and prove Theorem 1.7 - 1.9.

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## 2. BASIC FORMULAS FOR A PERTURBED METRIC

Let  $\Omega$  be an  $n$ -dimensional, smooth, compact manifold with boundary  $\Sigma$ . Let  $\bar{g}$  be a fixed smooth Riemannian metric on  $\Omega$ . Given a tensor  $\eta$ , let “ $|\eta|$ ” denote the length of  $\eta$  measured with respect to  $\bar{g}$ . Denote the covariant derivative with respect to  $\bar{g}$  by  $\bar{\nabla}$ . Indices of tensors are raised by  $\bar{g}$ . Let  $\bar{R}_{ikjl}$  denote the curvature tensor of  $\bar{g}$  such that if  $\bar{g}$  has constant sectional curvature  $\kappa$ , then  $\bar{R}_{ikjl} = \kappa(g_{ij}g_{kl} - g_{il}g_{kj})$ . Consider a nearby Riemannian metric  $g = \bar{g} + h$  where  $h$  is a symmetric  $(0, 2)$  tensor with  $|h|$  very small, say  $|h| \leq \frac{1}{2}$ .

The following pointwise estimates of the scalar curvature of  $g$  and the mean curvature of  $\Sigma$  were derived by Brendle and Marques in [5].

**Proposition 2.1** (Brendle and Marques [5]). *The scalar curvatures  $R(g)$ ,  $R(\bar{g})$  of the metrics  $g$ ,  $\bar{g}$  satisfy*

$$\begin{aligned} & |R(g) - R(\bar{g}) + \langle \text{Ric}(\bar{g}), h \rangle - \langle \text{Ric}(\bar{g}), h^2 \rangle \\ & + \frac{1}{4} |\bar{\nabla} h|^2 - \frac{1}{2} \bar{g}^{ij} \bar{g}^{kl} \bar{g}^{pq} \bar{\nabla}_i h_{kp} \bar{\nabla}_l h_{jq} + \frac{1}{4} |\bar{\nabla}(\text{tr}_{\bar{g}} h)|^2 \\ & + \bar{\nabla}_i [g^{ik} g^{jl} (\bar{\nabla}_k h_{jl} - \bar{\nabla}_l h_{jk})]| \\ & \leq C (|h| |\bar{\nabla} h|^2 + |h|^3) \end{aligned}$$

where  $\text{Ric}(\bar{g})$  is the Ricci curvature of  $\bar{g}$ ,  $h^2$  is the  $\bar{g}$ -square of  $h$ , i.e.  $(h^2)_{ik} = \bar{g}^{jl} h_{ij} h_{kl}$ ,  $\langle \cdot, \cdot \rangle$  is taken with respect to  $\bar{g}$ , and  $C$  is a positive constant depending only on  $n$ .

*Remark 2.1.* If the background metric  $\bar{g}$  is Ricci flat, i.e.  $\bar{R}_{ik} = 0$ , then there will be no  $|h|^3$  term in the above estimate. That is because

$$R(g) = g^{ik} \bar{R}_{ik} - g^{ik} g^{lj} (\bar{\nabla}_{i,k} h_{jl} - \bar{\nabla}_{i,l} h_{jk}) + g^{ik} g^{jl} g_{pq} (\Gamma_{il}^q \Gamma_{jk}^p - \Gamma_{jl}^q \Gamma_{ik}^p),$$

where each term on the right, except  $g^{ik} \bar{R}_{ik}$ , involves derivatives of  $h$ .

**Proposition 2.2** (Brendle and Marques [5]). *Assume that  $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$ , i.e.  $h|_{T\Sigma} = 0$  where  $T\Sigma$  is the tangent bundle of  $\Sigma$ . Then the mean curvatures  $H(g)$ ,  $H(\bar{g})$  of  $\Sigma$  in  $(\Omega, g)$ ,*

$(\Omega, \bar{g})$ , each with respect to the outward normals, satisfy

$$\begin{aligned} & \left| 2[H(g) - H(\bar{g})] - \left( h(\bar{\nu}, \bar{\nu}) - \frac{1}{4}h(\bar{\nu}, \bar{\nu})^2 + \sum_{\alpha=1}^{n-1} h(e_\alpha, \bar{\nu})^2 \right) H(\bar{g}) \right. \\ & \quad \left. + \left( 1 - \frac{1}{2}h(\bar{\nu}, \bar{\nu}) \right) \sum_{\alpha=1}^{n-1} [2\bar{\nabla}_{e_\alpha} h(e_\alpha, \bar{\nu}) - \bar{\nabla}_{\bar{\nu}} h(e_\alpha, e_\alpha)] \right| \\ & \leq C(|h|^2|\bar{\nabla}h| + |h|^3) \end{aligned}$$

where  $\{e_\alpha \mid 1 \leq \alpha \leq n-1\}$  is a local orthonormal frame on  $\Sigma$ ,  $\bar{\nu}$  is the  $\bar{g}$ -unit outward normal vector to  $\Sigma$ , and  $C$  is a positive constant depending only on  $n$ .

To derive the main formula (2.23) in this section, we let

$$(2.1) \quad DR_{\bar{g}}(h) = -\Delta_{\bar{g}}(\text{tr}_{\bar{g}}h) + \text{div}_{\bar{g}}\text{div}_{\bar{g}}h - \langle \text{Ric}(\bar{g}), h \rangle$$

be the linearization of the scalar curvature at  $\bar{g}$  along  $h$ . Here “ $\Delta_{\bar{g}}$ ,  $\text{div}_{\bar{g}}$ ” denote the Laplacian, the divergence with respect to  $\bar{g}$ .

**Lemma 2.1.** *With the same notations in Proposition 2.1, assume in addition  $\text{div}_{\bar{g}}h = 0$ , then*

$$\begin{aligned} R(g) - R(\bar{g}) &= DR_{\bar{g}}(h) - \frac{1}{2}DR_{\bar{g}}(h^2) + \langle h, \bar{\nabla}^2 \text{tr}_{\bar{g}}h \rangle - \frac{1}{4}(|\bar{\nabla}h|^2 + |\bar{\nabla}(\text{tr}_{\bar{g}}h)|^2) \\ &\quad + \frac{1}{2}h^{ij}h^{kl}\bar{R}_{ikjl} + E(h) + \bar{\nabla}_i(E_1^i(h)) \end{aligned}$$

where  $E(h)$  is a function and  $E_1(h)$  is a vector field on  $\Omega$  satisfying

$$|E(h)| \leq C(|h||\bar{\nabla}h|^2 + |h|^3), \quad |E_1(h)| \leq C|h|^2|\bar{\nabla}h|$$

for a positive constant  $C$  depending only on  $n$ .

*Proof.* First note that

$$(2.2) \quad -\bar{\nabla}_i [\bar{g}^{ik}\bar{g}^{jl}(\bar{\nabla}_k h_{jl} - \bar{\nabla}_l h_{jk})] - \langle \text{Ric}(\bar{g}), h \rangle = DR_{\bar{g}}(h).$$

Suppose  $g^{ik} = \bar{g}^{ik} + \tau^{ik}$ . Then  $\tau^{ik} = -h^{ik} + E_2^{ik}(h)$  where  $h^{ik} = \bar{g}^{ij}h_{jl}\bar{g}^{lk}$  and  $|E_2(h)| \leq C|h|^2$ . Hence,

$$g^{ik}g^{jl} - \bar{g}^{ik}\bar{g}^{jl} = -\bar{g}^{ik}h^{jl} - \bar{g}^{jl}h^{ik} + E_3^{ikjl}(h)$$

where  $|E_3(h)| \leq C|h|^2$ . Therefore,

$$\begin{aligned} (2.3) \quad & -\bar{\nabla}_i [(g^{ik}g^{jl} - \bar{g}^{ik}\bar{g}^{jl})(\bar{\nabla}_k h_{jl} - \bar{\nabla}_l h_{jk})] \\ & = \bar{\nabla}_i [(\bar{g}^{ik}h^{jl} + \bar{g}^{jl}h^{ik} - E_3^{ikjl}(h))(\bar{\nabla}_k h_{jl} - \bar{\nabla}_l h_{jk})] \\ & = \frac{1}{2}\Delta_{\bar{g}}|h|^2 + \langle h, \nabla^2 \text{tr}_{\bar{g}}(h) \rangle_{\bar{g}} - \text{div}_{\bar{g}}\text{div}_{\bar{g}}(h^2) - \bar{\nabla}_i (E_3^{ikjl}(\bar{\nabla}_k h_{jl} - \bar{\nabla}_l h_{jk})). \end{aligned}$$

Applying the Ricci identity, one has

$$(2.4) \quad \frac{1}{2} \bar{g}^{ij} \bar{g}^{kl} \bar{g}^{pq} \bar{\nabla}_i h_{kp} \bar{\nabla}_l h_{jq} = \frac{1}{2} \operatorname{div}_{\bar{g}} \operatorname{div}_{\bar{g}}(h^2) - \frac{1}{2} \langle \operatorname{Ric}(\bar{g}), h^2 \rangle + \frac{1}{2} h^{ij} h^{kl} \bar{R}_{ikjl}.$$

The lemma follows from Proposition 2.1, (2.2), (2.3) and (2.4).  $\square$

Next, let  $DH_{\bar{g}}(h)$  denote the linearization of the mean curvature at  $\bar{g}$  along  $h$ . Proposition 2.2 implies

$$(2.5) \quad DH_{\bar{g}}(h) = \frac{1}{2} \left[ h(\bar{\nu}, \bar{\nu}) H(\bar{g}) - \sum_{\alpha=1}^{n-1} (2 \bar{\nabla}_{e_\alpha} h(e_\alpha, \bar{\nu}) - \bar{\nabla}_{\bar{\nu}} h(e_\alpha, e_\alpha)) \right].$$

For later use, we note the following equivalent expression of  $DH_{\bar{g}}(h)$  (see [13, (34)] for instance)

$$(2.6) \quad DH_{\bar{g}}(h) = \frac{1}{2} \{ [d(\operatorname{tr}_{\bar{g}} h) - \operatorname{div}_{\bar{g}} h](\bar{\nu}) - \operatorname{div}_{\Sigma} X \},$$

where  $X$  is the vector field on  $\Sigma$  dual to the 1-form  $h(\bar{\nu}, \cdot)|_{T\Sigma}$ .

Let  $DR_{\bar{g}}^*(\cdot)$  denote the formal  $L^2$   $\bar{g}$ -adjoint of  $DR_{\bar{g}}(\cdot)$ , i.e.

$$(2.7) \quad DR_{\bar{g}}^*(\lambda) = -(\Delta_{\bar{g}} \lambda) \bar{g} + \nabla_{\bar{g}}^2 \lambda - \lambda \operatorname{Ric}(\bar{g})$$

where  $\lambda$  is a function and  $\nabla_{\bar{g}}^2 \lambda$  denotes the Hessian of  $\lambda$  with respect to  $\bar{g}$ . The content of the following lemma had been used in [13].

**Lemma 2.2.** *Let  $p$  be any smooth  $(0, 2)$  symmetric tensor on  $\Omega$ , then*

$$(2.8) \quad \begin{aligned} \int_{\Omega} DR_{\bar{g}}(p) \lambda \, d\operatorname{vol}_{\bar{g}} &= \int_{\Omega} \langle DR_{\bar{g}}^*(\lambda), p \rangle \, d\operatorname{vol}_{\bar{g}} - \int_{\Sigma} 2DH_{\bar{g}}(p) \lambda \, d\sigma_{\bar{g}} \\ &\quad + \int_{\Sigma} \lambda_{\bar{\nu}} (\operatorname{tr}_{\bar{g}}(p) - p(\bar{\nu}, \bar{\nu})) \, d\sigma_{\bar{g}} \end{aligned}$$

where  $\lambda_{\bar{\nu}} = \partial_{\bar{\nu}} \lambda$  denotes the directional derivative of  $\lambda$  along  $\bar{\nu}$ .

*Proof.* Let  $Y$  be the vector field on  $\Sigma$  dual to the 1-form  $p(\bar{\nu}, \cdot)|_{T\Sigma}$ . Integrating by parts, one has

$$\begin{aligned}
(2.9) \quad & \int_{\Omega} DR_{\bar{g}}(p)\lambda \, d\text{vol}_{\bar{g}} - \int_{\Omega} \langle DR_{\bar{g}}^*(\lambda), p \rangle \, d\text{vol}_{\bar{g}} \\
&= \int_{\Sigma} -\lambda \partial_{\bar{\nu}}(\text{tr}_{\bar{g}}p) + (\text{tr}_{\bar{g}}p) \partial_{\bar{\nu}}\lambda + \lambda \text{div}_{\bar{g}}p(\bar{\nu}) - p(\bar{\nu}, \bar{\nabla}^{\Sigma}\lambda) \, d\sigma_{\bar{g}} \\
&= \int_{\Sigma} \lambda [-\partial_{\bar{\nu}}(\text{tr}_{\bar{g}}p) + \text{div}_{\bar{g}}p(\bar{\nu})] - \langle Y, \bar{\nabla}^{\Sigma}\lambda \rangle \, d\sigma_{\bar{g}} + \int_{\Sigma} \lambda_{\bar{\nu}}(\text{tr}_{\bar{g}}(p) - p(\bar{\nu}, \bar{\nu})) \, d\sigma_{\bar{g}} \\
&= \int_{\Sigma} \lambda [-\partial_{\bar{\nu}}(\text{tr}_{\bar{g}}p) + \text{div}_{\bar{g}}p(\bar{\nu}) + \text{div}_{\Sigma}Y] \, d\sigma_{\bar{g}} + \int_{\Sigma} \lambda_{\bar{\nu}}(\text{tr}_{\bar{g}}(p) - p(\bar{\nu}, \bar{\nu})) \, d\sigma_{\bar{g}}
\end{aligned}$$

where  $\bar{\nabla}^{\Sigma}(\cdot)$  denotes the gradient on  $\Sigma$  with respect to the induced metric. From this and (2.6) the Lemma follows.  $\square$

Using Lemma 2.2, we can estimate  $\int_{\Omega} [R(g) - R(\bar{g})]\lambda \, d\text{vol}_{\bar{g}}$ .

**Proposition 2.3.** *Suppose  $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$  and  $h$  satisfies  $\text{div}_{\bar{g}}h = 0$ . Given any  $C^2$  function  $\lambda$  on  $\Omega$ , one has*

$$\begin{aligned}
& \int_{\Omega} [R(g) - R(\bar{g})]\lambda \, d\text{vol}_{\bar{g}} \\
&= \int_{\Omega} \langle h, DR_{\bar{g}}^*(\lambda) \rangle \, d\text{vol}_{\bar{g}} - \frac{1}{2} \int_{\Omega} \langle h^2, DR_{\bar{g}}^*(\lambda) \rangle \, d\text{vol}_{\bar{g}} \\
& \quad + \int_{\Omega} \left[ (\text{tr}_{\bar{g}}h) \langle h, \nabla_{\bar{g}}^2\lambda \rangle + \frac{1}{2} h^{ij} h^{kl} \bar{R}_{ikjl} \lambda - \frac{1}{4} (|\bar{\nabla}h|^2 + |\bar{\nabla}(\text{tr}_{\bar{g}}h)|^2) \lambda \right] \, d\text{vol}_{\bar{g}} \\
& \quad + \int_{\Sigma} \left[ -(h_{nn})^2 - \frac{1}{2} |X|^2 \right] \lambda_{;n} \, d\sigma_{\bar{g}} - \int_{\Sigma} h_{nn} \langle X, \bar{\nabla}^{\Sigma}\lambda \rangle \, d\sigma_{\bar{g}} \\
& \quad + \int_{\Sigma} \left[ -\frac{1}{2} (h_{nn})^2 H(\bar{g}) - \frac{1}{2} \bar{\mathbb{I}}\mathbb{I}(X, X) - \frac{3}{2} |X|^2 H(\bar{g}) \right] \lambda \, d\sigma_{\bar{g}} - \int_{\Sigma} (2 - 2\text{tr}_{\bar{g}}h) DH_{\bar{g}}(h) \lambda \, d\sigma_{\bar{g}} \\
& \quad + \int_{\Omega} E(h) \lambda \, d\text{vol}_{\bar{g}} - \int_{\Omega} E_1^i(h) \bar{\nabla}_i \lambda \, d\text{vol}_{\bar{g}} + \int_{\Sigma} F_1(h) \lambda \, d\sigma_{\bar{g}}
\end{aligned}$$

where  $\bar{\mathbb{I}}\mathbb{I}$  is the second fundamental form of  $\Sigma$  in  $(\Omega, \bar{g})$  with respect to  $\bar{\nu}$ ,  $X$  is the vector field on  $\Sigma$  that is dual to the 1-form  $h(e_n, \cdot)|_{T\Sigma}$ ,  $E(h)$  and  $E_1^i(h)$  are as in Lemma 2.1, and  $F_1(h)$  is a function on  $\Sigma$  satisfying

$$|F_1(h)| \leq C|h|^2|\bar{\nabla}h|$$

for a positive constant  $C$  depending only on  $n$ .



*Proof.* By (2.8) with  $p = h$ , using the fact that  $h|_{T(\Sigma)} = 0$ , we have

$$(2.10) \quad \int_{\Omega} DR_{\bar{g}}(h)\lambda \, d\text{vol}_{\bar{g}} = \int_{\Omega} \langle DR_{\bar{g}}^*(\lambda), h \rangle \, d\text{vol}_{\bar{g}} - \int_{\Sigma} 2DH_{\bar{g}}(h)\lambda \, d\sigma_{\bar{g}}.$$

By the second line in (2.9) with  $p = h^2$ , and integrating by parts, we also have

$$(2.11) \quad \begin{aligned} & \int_{\Omega} -\frac{\lambda}{2} DR_{\bar{g}}(h^2) + \lambda \langle h, \bar{\nabla}^2 \text{tr}_{\bar{g}} h \rangle \, d\text{vol}_{\bar{g}} \\ &= \int_{\Omega} -\frac{1}{2} \langle DR_{\bar{g}}^*(\lambda), h^2 \rangle + \text{tr}_{\bar{g}} h \langle h, \bar{\nabla}^2 \lambda \rangle \, d\text{vol}_{\bar{g}} + \mathcal{B} \end{aligned}$$

where

$$(2.12) \quad \begin{aligned} \mathcal{B} &= \int_{\Sigma} \frac{1}{2} [\lambda \partial_{\bar{\nu}}(|h|^2) - |h|^2 \partial_{\bar{\nu}} \lambda - \lambda (\text{div}_{\bar{g}} h^2)(\bar{\nu}) + (h^2)(\bar{\nu}, \bar{\nabla} \lambda)] \, d\sigma_{\bar{g}} \\ &+ \int_{\Sigma} [\lambda h(\bar{\nu}, \bar{\nabla} \text{tr}_{\bar{g}} h) - \text{tr}_{\bar{g}} h h(\bar{\nu}, \bar{\nabla} \lambda)] \, d\sigma_{\bar{g}}. \end{aligned}$$

To compute  $\mathcal{B}$ , let  $\{e_{\alpha} \mid 1 \leq \alpha \leq n-1\}$  be an orthonormal frame on  $\Sigma$  and let  $e_n = \bar{\nu}$ . Denote  $\bar{\nabla}$  also by “ $\bar{\nabla}$ ”, thus  $h_{ij;k} = \bar{\nabla}_k h_{ij}$ . The assumptions  $h|_{T\Sigma} = 0$  and  $\text{div}_{\bar{g}} h = 0$  imply the following facts on  $\Sigma$ :

$$(2.13) \quad |h|^2 = (h_{nn})^2 + 2|X|^2, \quad (h^2)_{nn} = (h_{nn})^2 + |X|^2, \quad (h^2)_{n\alpha} = h_{nn}h_{n\alpha},$$

$$(2.14) \quad (h^2)(\bar{\nu}, \bar{\nabla} \lambda) = [(h_{nn})^2 + |X|^2] \lambda_{;n} + h_{nn} \langle X, \bar{\nabla}^{\Sigma} \lambda \rangle,$$

$$(2.15) \quad h_{\beta\gamma;\alpha} = h_{\beta n} \bar{\mathbb{I}}\bar{\mathbb{I}}_{\gamma\alpha} + h_{n\gamma} \bar{\mathbb{I}}\bar{\mathbb{I}}_{\beta\alpha},$$

$$(2.16) \quad h_{nn;\alpha} = (\text{tr}_{\bar{g}} h)_{;\alpha} - \sum_{\beta=1}^{n-1} h_{\beta\beta;\alpha} = (\text{tr}_{\bar{g}} h)_{;\alpha} - 2\bar{\mathbb{I}}\bar{\mathbb{I}}(X, e_{\alpha}),$$

$$(2.17) \quad 0 = (\text{div} h)_{\alpha} = h_{\alpha n;n} + \sum_{\beta=1}^{n-1} h_{\alpha\beta;\beta} = h_{\alpha n;n} + h_{n\alpha} H(\bar{g}) + \bar{\mathbb{I}}\bar{\mathbb{I}}(X, e_{\alpha}),$$

$$(2.18)$$

$$0 = (\text{div}_{\bar{g}} h)_n = h_{nn;n} + \sum_{\alpha=1}^{n-1} h_{n\alpha;\alpha} = h_{nn;n} + \text{div}_{\Sigma} X + h_{nn} H(\bar{g}),$$

$$(2.19) \quad 2DH_{\bar{g}}(h) = (\text{tr}_{\bar{g}} h)_{;n} - \text{div}_{\Sigma} X,$$

where (2.19) follows from (2.6). By (2.16)-(2.18), we have

$$\begin{aligned}
 (2.20) \quad & \partial_{\bar{\nu}}(|h|^2) - (\operatorname{div}_{\bar{g}} h^2)(\bar{\nu}) \\
 &= 3h_{n\alpha}h_{n\alpha;n} + h_{nn}h_{nn;n} - h_{n\alpha}h_{nn;\alpha} \\
 &= -\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) - 3H(\bar{g})|X|^2 - H(\bar{g})(h_{nn})^2 - h_{nn}\operatorname{div}_{\Sigma}X - \langle X, \bar{\nabla}^{\Sigma}\operatorname{tr}_{\bar{g}}h \rangle.
 \end{aligned}$$

By (2.12), (2.13), (2.14), (2.20) and integration by parts, we have

$$\begin{aligned}
 (2.21) \quad \mathcal{B} &= \int_{\Sigma} \left[ -(h_{nn})^2 - \frac{1}{2}|X|^2 \right] \lambda_{;n} - \int_{\Sigma} h_{nn} \langle X, \bar{\nabla}^{\Sigma} \lambda \rangle \\
 &\quad + \int_{\Sigma} \left[ -\frac{1}{2}\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) - \frac{3}{2}H(\bar{g})|X|^2 - \frac{1}{2}H(\bar{g})(h_{nn})^2 + 2h_{nn}DH_{\bar{g}}(h) \right] \lambda d\sigma_{\bar{g}}.
 \end{aligned}$$

Note that

$$(2.22) \quad \int_{\Omega} (\bar{\nabla}_i E_1^i(h)) \lambda \, d\operatorname{vol}_{\bar{g}} = - \int_{\Omega} E_1^i(h) \bar{\nabla}_i \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} \lambda F_1(h) \, d\sigma_{\bar{g}}$$

where  $|F_1(h)| = \langle E_1(h), \bar{\nu} \rangle \leq C|h|^2|\bar{\nabla}h|$ . Proposition 2.3 now follows from Lemma 2.1, (2.10), (2.11), (2.21), and (2.22).  $\square$

The formula (2.23) next is a general form of [5, Theorem 10], which Brendle and Marques derived for geodesic balls in  $\mathbb{S}^n$ .

**Theorem 2.1.** *Suppose  $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$  and  $h$  satisfies  $\operatorname{div}_{\bar{g}}h = 0$ . Given any  $C^2$  function  $\lambda$  on  $\Omega$ , one has*

$$\begin{aligned}
 (2.23) \quad & \int_{\Omega} [R(g) - R(\bar{g})] \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} (2 - \operatorname{tr}_{\bar{g}}h) [H(g) - H(\bar{g})] \lambda \, d\sigma_{\bar{g}} \\
 &= \int_{\Omega} \langle h, DR_{\bar{g}}^*(\lambda) \rangle \, d\operatorname{vol}_{\bar{g}} - \frac{1}{2} \int_{\Omega} \langle h^2, DR_{\bar{g}}^*(\lambda) \rangle \, d\operatorname{vol}_{\bar{g}} \\
 &\quad + \int_{\Omega} \left[ (\operatorname{tr}_{\bar{g}}h) \langle h, \nabla_{\bar{g}}^2 \lambda \rangle + \frac{1}{2} h^{ij} h^{kl} \bar{R}_{ikjl} \lambda - \frac{1}{4} (|\bar{\nabla}h|^2 + |\bar{\nabla}(\operatorname{tr}_{\bar{g}}h)|^2) \lambda \right] \, d\operatorname{vol}_{\bar{g}} \\
 &\quad + \int_{\Sigma} \left[ -\frac{1}{4} (h_{nn})^2 H(\bar{g}) - \frac{1}{2} (\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) + H(\bar{g})|X|^2) \right] \lambda \, d\sigma_{\bar{g}} \\
 &\quad + \int_{\Sigma} \lambda_{;n} \left[ -(h_{nn})^2 - \frac{1}{2}|X|^2 \right] \, d\sigma_{\bar{g}} + \int_{\Sigma} (-1) h_{nn} \langle X, \bar{\nabla}^{\Sigma} \lambda \rangle \, d\sigma_{\bar{g}} \\
 &\quad + \int_{\Omega} E(h) \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Omega} Z^i(h) \bar{\nabla}_i \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} F(h) \lambda \, d\sigma_{\bar{g}}
 \end{aligned}$$

where  $E(h)$  is a function and  $Z(h)$  is a vector field on  $\Omega$  satisfying

$$|E(h)| \leq C(|h||\bar{\nabla}h|^2 + |h|^3), \quad |Z(h)| \leq C|h|^2|\bar{\nabla}h|,$$

and  $F(h)$  is some function on  $\Sigma$  satisfying

$$|F(h)| \leq C(|h|^2|\bar{\nabla}h| + |h|^3).$$

*Proof.* Proposition 2.2 implies

$$(2.24) \quad 2[H(g) - H(\bar{g})] = 2DH_{\bar{g}}(h) + J(h) + F_2(h)$$

where

$$J(h) = \left[ \frac{1}{4}(h_{nn})^2 + |X|^2 \right] H(\bar{g}) - h_{nn}DH_{\bar{g}}(h)$$

and  $F_2(h)$  is some function on  $\Sigma$  satisfying  $|F_2(h)| \leq C(|h|^2|\bar{\nabla}h| + |h|^3)$ .

Therefore

$$(2.25) \quad \begin{aligned} & (2 - h_{nn})[H(g) - H(\bar{g})] \\ &= (2 - 2h_{nn})DH_{\bar{g}}(h) + \left[ \frac{1}{4}(h_{nn})^2 + |X|^2 \right] H(\bar{g}) \\ & \quad + F_2(h) - \frac{1}{2}h_{nn}[J(h) + F_2(h)]. \end{aligned}$$

(2.23) now follows readily from Proposition 2.3 and (2.25).  $\square$

The term  $DR_{\bar{g}}^*(\lambda)$  in (2.23) may suggest that one consider a background metric  $\bar{g}$  which admits a nontrivial function  $\lambda$  such that  $DR_{\bar{g}}^*(\lambda) = 0$  (such metrics are known as *static metrics* [10].) For instance, if  $\bar{g}$  is the standard metric on  $\mathbb{S}^n$  and  $\lambda = \cos r$ , where  $r$  is the  $\bar{g}$ -distance to a point, then (2.23) reduces to the formula in [5, Theorem 10].

Besides static metrics, one can also consider those metrics  $\bar{g}$  with the property that there exists a function  $\lambda$  such that

$$(2.26) \quad DR_{\bar{g}}^*(\lambda) = \bar{g}.$$

These metrics were studied by the authors in [13] and [14]. In this case, the terms

$$\int_{\Omega} \langle h, DR_{\bar{g}}^*(\lambda) \rangle d\text{vol}_{\bar{g}} - \frac{1}{2} \int_{\Omega} \langle h^2, DR_{\bar{g}}^*(\lambda) \rangle d\text{vol}_{\bar{g}}$$

in (2.23) become

$$\int_{\Omega} \text{tr}_{\bar{g}} h d\text{vol}_{\bar{g}} - \frac{1}{2} \int_{\Omega} |h|^2 d\text{vol}_{\bar{g}}.$$

To compensate these terms, one can include the difference between the volumes of  $g$  and  $\bar{g}$  into (2.23).

**Corollary 2.1.** *Suppose  $\bar{g}$  is a metric on  $\Omega$  with the property that there exists a function  $\lambda$  satisfying  $DR_{\bar{g}}^*(\lambda) = \bar{g}$ . Let  $g = \bar{g} + h$  be a nearby metric such that  $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$  and  $h$  satisfies  $\operatorname{div}_{\bar{g}} h = 0$ . Let  $V(g)$ ,  $V(\bar{g})$  denote the volume of  $(\Omega, g)$ ,  $(\Omega, \bar{g})$ . Then*

$$\begin{aligned}
(2.27) \quad & -2(V(g) - V(\bar{g})) + \int_{\Omega} [R(g) - R(\bar{g})] \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} (2 - \operatorname{tr}_{\bar{g}} h) [H(g) - H(\bar{g})] \lambda \, d\sigma_{\bar{g}} \\
&= \int_{\Omega} \left[ -\frac{1}{4} - \frac{1}{n-1} \right] (\operatorname{tr}_{\bar{g}} h)^2 \, d\operatorname{vol}_{\bar{g}} + \int_{\Omega} \left[ -\frac{1}{4} (|\bar{\nabla} h|^2 + |\nabla_{\bar{g}}(\operatorname{tr}_{\bar{g}} h)|^2) \lambda \right] \, d\operatorname{vol}_{\bar{g}} \\
&+ \int_{\Omega} \left[ \frac{1}{1-n} R(\bar{g}) (\operatorname{tr}_{\bar{g}} h)^2 + \langle h, \operatorname{Ric}(\bar{g}) \rangle (\operatorname{tr}_{\bar{g}} h) + \frac{1}{2} h_{ij} h_{kl} R_{ikjl} \right] \lambda \, d\operatorname{vol}_{\bar{g}} \\
&+ \int_{\Sigma} \left[ -\frac{1}{4} (h_{nn})^2 H(\bar{g}) - \frac{1}{2} (\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) + H(\bar{g}) |X|^2) \right] \lambda \, d\sigma_{\bar{g}} \\
&+ \int_{\Sigma} \lambda_{;n} \left[ -(h_{nn})^2 - \frac{1}{2} |X|^2 \right] \, d\sigma_{\bar{g}} + \int_{\Sigma} (-1) h_{nn} \langle X, \bar{\nabla}^{\Sigma} \lambda \rangle \, d\sigma_{\bar{g}} \\
&+ \int_{\Omega} G(h) \, d\operatorname{vol}_{\bar{g}} + \int_{\Omega} E(h) \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Omega} Z^i(h) \bar{\nabla}_i \lambda \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} F(h) \lambda \, d\sigma_{\bar{g}}
\end{aligned}$$

where  $G(h)$  and  $E(h)$  are functions on  $\Omega$  satisfying

$$|G(h)| \leq C|h|^3, \quad |E(h)| \leq C(|h| |\bar{\nabla} h|^2 + |h|^3),$$

$Z(h)$  is a vector field on  $\Omega$  satisfying

$$|Z(h)| \leq C|h|^2 |\bar{\nabla} h|,$$

and  $F(h)$  is a function on  $\Sigma$  satisfying

$$|F(h)| \leq C(|h|^2 |\bar{\nabla} h| + |h|^3).$$

*Proof.* The difference between the volumes of  $\bar{g}$  and  $g = \bar{g} + h$  is

$$(2.28) \quad V(g) - V(\bar{g}) = \int_{\Omega} \frac{1}{2} (\operatorname{tr}_{\bar{g}} h) + \left[ \frac{1}{8} (\operatorname{tr}_{\bar{g}} h)^2 - \frac{1}{4} |h|^2 \right] + G(h) \, d\operatorname{vol}_{\bar{g}},$$

where  $G(h)$  is a function satisfying  $|G(h)| \leq C|h|^3$  for a constant  $C$  depending only on  $n$ . Suppose  $DR_{\bar{g}}^*(\lambda) = \bar{g}$ , i.e.

$$-(\Delta_{\bar{g}} \lambda) \bar{g} + \nabla_{\bar{g}}^2 \lambda - \lambda \operatorname{Ric}(\bar{g}) = \bar{g}.$$

Taking trace, one has  $\Delta_{\bar{g}} \lambda = \frac{1}{1-n} [R(\bar{g}) \lambda + n]$ . Thus,

$$(2.29) \quad \nabla_{\bar{g}}^2 \lambda = \frac{1}{1-n} [R(\bar{g}) \lambda + 1] \bar{g} + \lambda \operatorname{Ric}(\bar{g}).$$

(2.27) follows from (2.23), (2.28) and (2.29).  $\square$

## 3. VOLUME CONSTRAINED RIGIDITY

We prove Theorem 1.3 in this section. First, we recall its statement:

**Theorem 3.1.** *Let  $(\Omega, \bar{g})$  be an  $n$ -dimensional compact Riemannian manifold, of constant sectional curvature 1, with smooth boundary  $\Sigma$ . Suppose  $\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$  (i.e.  $\bar{\mathbb{I}} + \bar{H}\bar{\gamma}$  is positive semi-definite), where  $\bar{\gamma}$  is the induced metric on  $\Sigma$  and  $\bar{\mathbb{I}}, \bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$  in  $(\Omega, \bar{g})$ . Suppose the first nonzero Neumann eigenvalue  $\mu$  of  $(\Omega, g)$  satisfies  $\mu > n - \frac{2}{n+1}$ .*

*Consider a nearby metric  $g$  on  $\Omega$  with the properties*

- $R(g) \geq n(n-1)$  where  $R(g)$  is the scalar curvature of  $g$
- $H(g) \geq \bar{H}$  where  $H(g)$  is the mean curvature of  $\Sigma$  in  $(\Omega, g)$
- $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$
- $V(g) \geq V(\bar{g})$  where  $V(g), V(\bar{g})$  are the volumes of  $g, \bar{g}$ .

*If  $\|g - \bar{g}\|_{C^2(\bar{\Omega})}$  is sufficiently small, then there is a diffeomorphism  $\varphi$  on  $\Omega$  with  $\varphi|_{\Sigma} = \text{id}$ , which is the identity map on  $\Sigma$ , such that  $\varphi^*(g) = \bar{g}$ .*

*Proof.* Fix a real number  $p > n$ . By [5, Proposition 11], if  $\|g - \bar{g}\|_{W^{2,p}(\Omega)}$  is sufficiently small, there exists a  $W^{3,p}$  diffeomorphism  $\varphi$  on  $\Omega$  with  $\varphi|_{\Sigma} = \text{id}$  such that  $h = \varphi^*(g) - \bar{g}$  is divergence free with respect to  $\bar{g}$ , and  $\|h\|_{W^{2,p}(\Omega)} \leq N\|g - \bar{g}\|_{W^{2,p}(\Omega)}$  for some positive constant  $N$  depending only on  $(\Omega, \bar{g})$ . Replacing  $g$  by  $\varphi^*(g)$ , we may assume  $g = \bar{g} + h$  with  $\text{div}_{\bar{g}} h = 0$ . We want to prove that if  $\|h\|_{C^1(\bar{\Omega})}$  is sufficiently small and  $g$  satisfies the conditions in the theorem, then  $h$  must be zero.

Since  $\bar{g}$  has constant sectional curvature 1, we choose  $\lambda = -\frac{1}{n-1}$  such that  $DR_{\bar{g}}^*(\lambda) = \bar{g}$ . Corollary 2.1 then shows

$$\begin{aligned}
 (3.1) \quad & -2(V(g) - V(\bar{g})) - \frac{1}{n-1} \int_{\Omega} [R(g) - R(\bar{g})] \, d\text{vol}_{\bar{g}} \\
 & - \frac{1}{n-1} \int_{\Sigma} (2 - \text{tr}_{\bar{g}} h) [H(g) - H(\bar{g})] \, d\sigma_{\bar{g}} \\
 \geq & \frac{1}{4(n-1)} \int_{\Omega} [-(n+1)(\text{tr}_{\bar{g}} h)^2 + 2|h|^2 + |\bar{\nabla} h|^2 + |\bar{\nabla}(\text{tr}_{\bar{g}} h)|^2] \, d\text{vol}_{\bar{g}} \\
 & + \frac{1}{4(n-1)} \int_{\Sigma} [(h_{nn})^2 H(\bar{g}) + 2(\bar{\mathbb{I}}(X, X) + H(\bar{g})|X|^2)] \, d\sigma_{\bar{g}} \\
 & - C\|h\|_{C^1(\bar{\Omega})} \left[ \int_{\Omega} (|h|^2 + |\bar{\nabla} h|^2) \, d\text{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 \, d\sigma_{\bar{g}} \right]
 \end{aligned}$$

for a constant  $C$  depending only on  $(\Omega, \bar{g})$ .

Using the variational property of  $\mu$ , we have

$$(3.2) \quad \int_{\Omega} |\bar{\nabla}(\text{tr}_{\bar{g}}h)|^2 d\text{vol}_{\bar{g}} \geq \mu \left[ \left( \int_{\Omega} (\text{tr}_{\bar{g}}h)^2 d\text{vol}_{\bar{g}} \right) - \frac{1}{V(\bar{g})} \left( \int_{\Omega} \text{tr}_{\bar{g}}h d\text{vol}_{\bar{g}} \right)^2 \right].$$

By (2.28),  $\int_{\Omega} \text{tr}_{\bar{g}}h d\text{vol}_{\bar{g}}$  is related to  $(V(g) - V(\bar{g}))$  by

$$(3.3) \quad \int_{\Omega} \text{tr}_{\bar{g}}h d\text{vol}_{\bar{g}} = 2(V(g) - V(\bar{g})) - \int_{\Omega} \left\{ \left[ \frac{1}{4}(\text{tr}_{\bar{g}}h)^2 - \frac{1}{2}|h|^2 \right] + 2G(h) \right\} d\text{vol}_{\bar{g}},$$

where  $G(h) \leq C|h|^3$ .

Given any constant  $0 < \epsilon < 1$ , using (3.2) and the fact  $|h|^2 \geq \frac{1}{n}(\text{tr}_{\bar{g}}h)^2$  and  $|\bar{\nabla}h|^2 \geq \frac{1}{n}|\bar{\nabla}(\text{tr}_{\bar{g}}h)|^2$ , we have

$$(3.4) \quad \begin{aligned} & \int_{\Omega} [-(n+1)(\text{tr}_{\bar{g}}h)^2 + 2|h|^2 + |\bar{\nabla}h|^2 + |\nabla_{\bar{g}}(\text{tr}_{\bar{g}}h)|^2] d\text{vol}_{\bar{g}} \\ & \geq \int_{\Omega} \left[ \epsilon|h|^2 + \epsilon|\bar{\nabla}h|^2 + \left[ -(n+1) + \frac{2-\epsilon}{n} \right] (\text{tr}_{\bar{g}}h)^2 + \left[ \frac{(1-\epsilon)}{n} + 1 \right] |\bar{\nabla}(\text{tr}_{\bar{g}}h)|^2 \right] d\text{vol}_{\bar{g}} \\ & \geq \int_{\Omega} \left[ \epsilon|h|^2 + \epsilon|\bar{\nabla}h|^2 + \left[ -(n+1) + \frac{2-\epsilon}{n} + \frac{(1-\epsilon)}{n}\mu + \mu \right] (\text{tr}_{\bar{g}}h)^2 \right] d\text{vol}_{\bar{g}} \\ & \quad - \mu \left[ \frac{(1-\epsilon)}{n} + 1 \right] \frac{1}{V(\bar{g})} \left( \int_{\Omega} \text{tr}_{\bar{g}}h d\text{vol}_{\bar{g}} \right)^2. \end{aligned}$$

Since  $\mu > n - \frac{2}{n+1}$ , we can chose  $\epsilon$  (depending only on  $\mu$  and  $n$ ) such that

$$(3.5) \quad \left[ -(n+1) + \frac{2-\epsilon}{n} + \frac{(1-\epsilon)}{n}\mu + \mu \right] \geq 0.$$

Then it follows from (3.3), (3.4) and (3.5) that

$$(3.6) \quad \begin{aligned} & \int_{\Omega} (-(n+1)(\text{tr}_{\bar{g}}h)^2 + 2|h|^2 + |\bar{\nabla}h|^2 + |\nabla_{\bar{g}}(\text{tr}_{\bar{g}}h)|^2) d\text{vol}_{\bar{g}} \\ & \geq \epsilon \int_{\Omega} (|h|^2 + |\bar{\nabla}h|^2) d\text{vol}_{\bar{g}} - C_1(V(g) - V(\bar{g}))^2 - C_1 \int_{\Omega} |h|^4 d\sigma_{\bar{g}} \end{aligned}$$

where  $C_1$  is a positive constant depending only on  $(\Omega, \bar{g})$ .

At the boundary  $\Sigma$ , the assumption  $\bar{\mathbb{I}}\bar{\mathbb{I}} + H(\bar{g})\bar{\gamma} \geq 0$  implies  $H(\bar{g}) \geq 0$ , therefore

$$(3.7) \quad \int_{\Sigma} [(h_{nn})^2 H(\bar{g}) + 2\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) + H(\bar{g})|X|^2] d\sigma_{\bar{g}} \geq 0$$

for any  $h$ . By (3.1), (3.6) and (3.7), we have

$$\begin{aligned}
 & -8(n-1)(V(g) - V(\bar{g})) - 4 \int_{\Omega} [R(g) - R(\bar{g})] \, d\text{vol}_{\bar{g}} \\
 & - 4 \int_{\Sigma} (2 - \text{tr}_{\bar{g}} h) [H(g) - H(\bar{g})] \, d\sigma_{\bar{g}} \\
 (3.8) \quad & \geq \epsilon \int_{\Omega} (|h|^2 + |\bar{\nabla} h|^2) \, d\text{vol}_{\bar{g}} \\
 & - C(V(g) - V(\bar{g}))^2 - C \int_{\Omega} |h|^4 \, d\text{vol}_{\bar{g}} \\
 & - C\|h\|_{C^1(\bar{\Omega})} \left[ \int_{\Omega} (|h|^2 + |\bar{\nabla} h|^2) \, d\text{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 \, d\sigma_{\bar{g}} \right]
 \end{aligned}$$

for some positive constant  $C$  depending only on  $(\Omega, \bar{g})$ .

Finally, we note that

$$(3.9) \quad (V(g) - V(\bar{g}))^2 \leq C \left( \int_{\Omega} |h| \, d\text{vol}_{\bar{g}} \right) (V(g) - V(\bar{g}))$$

by (3.3) and the assumption  $V(g) \geq V(\bar{g})$ . Also, by the trace theorem,

$$(3.10) \quad \|h\|_{L^2(\Sigma)} \leq C\|h\|_{W^{1,2}(\Omega)}$$

for a constant  $C$  only depending on  $\Omega$ . Therefore, by (3.8), (3.9), (3.10) and the assumptions  $V(g) \geq V(\bar{g})$ ,  $R(g) \geq R(\bar{g})$  and  $H(g) \geq H(\bar{g})$ , we conclude that if  $\|h\|_{C^1(\bar{\Omega})}$  is sufficiently small, then

$$(3.11) \quad 0 \geq \frac{\epsilon}{2} \int_{\Omega} (|h|^2 + |\bar{\nabla} h|^2) \, d\text{vol}_{\bar{g}}$$

which implies  $h$  must be identically zero. This completes the proof.  $\square$

*Remark 3.1.* In Theorem 3.1, if  $\Sigma$  is indeed empty, i.e.  $(\Omega, \bar{g})$  is a closed space form, its first nonzero Neumann eigenvalue satisfies  $\mu \geq n$  as  $(\Omega, \bar{g})$  is covered by  $\mathbb{S}^n$ . In this case, Theorem 3.1 says that  $V(g) \geq V(\bar{g})$  implies  $g$  is isometric to  $\bar{g}$  for a nearby metrics  $g$  with  $R(g) \geq R(\bar{g})$ . This could be compared to a more profound theorem known in 3-dimension: “If  $(M, g)$  is closed 3-manifold with  $R(g) \geq 6$ ,  $\text{Ric}(g) \geq g$  and  $V(g) \geq V(\mathbb{S}^3)$ , then  $(M, g)$  is isometric to  $\mathbb{S}^3$ .” (See [4, Corollary 5.4] and earlier reference of [3, 11])

When  $\Sigma \neq \emptyset$ , the boundary assumption  $\bar{\text{III}} + \bar{H}\bar{\gamma} \geq 0$  in Theorem 3.1 can be relaxed in certain circumstances. A detailed examination of the above proof shows, if

$$(3.12) \quad \bar{\text{III}}(v, v) + \bar{H}\bar{\gamma} \geq -\beta\bar{\gamma}$$

for some positive constant  $\beta$ , where  $\beta$  is sufficiently small comparing to the constant  $\epsilon$  in (3.5) and the constant  $C$  in (3.10), then the conclusion of Theorem 3.1 still holds on such an  $(\Omega, \bar{g})$ . In particular, this shows

**Corollary 3.1.** *Let  $(M, \bar{g})$  be an  $n$ -dimensional Riemannian manifold of constant sectional curvature 1. Suppose  $\Omega \subset M$  is a bounded domain with smooth boundary  $\Sigma$ , satisfying the assumptions in Theorem 3.1, i.e  $\mu > n - \frac{2}{n+1}$  and  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$  on  $\Sigma$ . Let  $\tilde{\Omega} \subset M$  be another bounded domain with smooth boundary  $\tilde{\Sigma}$ . If  $\tilde{\Sigma}$  is sufficiently close to  $\Sigma$  in the  $C^2$  norm, then the conclusion of Theorem 3.1 holds on  $\tilde{\Omega}$ .*

It is known that the first nonzero Neumann eigenvalue of  $\mathbb{S}_+^n$  is  $n$  (see [9, Theorem 3]). Therefore, Theorem 1.2 follows from Theorem 3.1. Moreover, by Corollary 3.1, Theorem 3.1 holds on a geodesic ball in  $\mathbb{S}^n$  whose radius is slightly larger than  $\frac{\pi}{2}$ .

By the next lemma, we know Theorem 3.1 also holds on any geodesic ball in  $\mathbb{S}^n$  that is strictly contained in  $\mathbb{S}_+^n$ .

**Lemma 3.1.** *Let  $B(\delta) \subset \mathbb{S}^n$  be a geodesic ball of radius  $\delta$ . Let  $\mu(\delta)$  be the first nonzero Neumann eigenvalue of  $B(\delta)$ .*

- (i)  $\mu(\delta)$  is a strictly decreasing function of  $\delta$  on  $(0, \frac{\pi}{2}]$ .
- (ii) For any  $0 < \delta < \frac{\pi}{2}$ ,

$$\mu(\delta) > n + \frac{(\sin \delta)^{n-2} \cos \delta}{\int_0^\delta (\sin t)^{n-1} dt} > \frac{n}{(\sin \delta)^2}.$$

*Proof.* By [9, Theorem 2, p.44],  $\mu(\delta)$  is characterized by the fact that

$$(3.13) \quad \{(\sin t)^{n-1} J'\}' + [\mu(\delta) - (n-1)(\sin t)^{-2}](\sin t)^{n-1} J = 0$$

has a solution  $J = J(t)$  on  $[0, \delta]$  satisfying

$$(3.14) \quad J(0) = 0, \quad J'(\delta) = 0, \quad J'(t) \neq 0, \quad \forall t \in [0, \delta).$$

Given  $0 < \delta_1 < \delta_2 \leq \frac{\pi}{2}$ , let  $J_i = J_i(t)$  be a solution to (3.13) with  $\mu(\delta)$  replaced by  $\mu(\delta_i)$ , satisfying (3.14) on  $[0, \delta_i]$ ,  $i = 1, 2$ . Replacing  $J_i$  by  $-J_i$  if necessary, we may assume that  $J'_i > 0$  on  $[0, \delta_i)$ , hence  $J_i > 0$  on  $(0, \delta_i]$ . Define

$$f_i = \frac{(\sin t)^{n-1} J'_i}{J_i}, \quad \beta_i(t) = \left[ \mu(\delta_i) - \frac{n-1}{(\sin t)^2} \right] (\sin t)^{n-1}.$$

By (3.13),  $f_i$  satisfies

$$f'_i = -\beta_i - \frac{1}{(\sin t)^{n-1}} f_i^2.$$



Therefore, on  $(0, \delta_1]$ ,

$$(3.15) \quad (f_1 - f_2)' = \frac{1}{(\sin t)^{n-1}}(f_2^2 - f_1^2) + [\mu(\delta_2) - \mu(\delta_1)](\sin t)^{n-1}.$$

Note that  $f_1(t)$ ,  $f_2(t)$  can be extended continuously to 0 such that  $f_1(0) = f_2(0)$ . Moreover,  $f_1 > 0$ ,  $f_2 > 0$  on  $(0, \delta_1)$ ,  $f_2(\delta_1) > 0 = f_1(\delta_1)$ . Let  $0 \leq t_0 < \delta_1$  be such that  $f_1 = f_2$  at  $t_0$  and  $f_2 > f_1$  for  $t_0 < t \leq \delta_1$ . On  $(t_0, \delta_1]$ , one would have  $(f_1 - f_2)' > 0$  if  $\mu(\delta_2) \geq \mu(\delta_1)$ , which is a contradiction to  $f_2 > f_1$ . Therefore,  $\mu(\delta_2) < \mu(\delta_1)$ . This proves (i).

To prove (ii), we further claim that  $t_0 = 0$ , i.e.  $f_2 > f_1$  on  $(0, \delta_1]$ . If not, there would be a nonpositive local minimum of  $(f_2 - f_1)$  at some  $\tilde{t}_0 \in (0, t_0]$ . At  $\tilde{t}_0$ , (3.15) implies

$$(3.16) \quad 0 = (f_1 - f_2)' \leq [\mu(\delta_2) - \mu(\delta_1)](\sin \tilde{t}_0)^{n-1} < 0$$

because  $0 < f_2(\tilde{t}_0) \leq f_1(\tilde{t}_0)$  and  $\mu(\delta_2) < \mu(\delta_1)$ . Hence  $f_2 > f_1$  on  $(0, \delta_1]$ . Integrating (3.15) on  $[0, \delta_1]$ , we have

$$(3.17) \quad -f_2(\delta_1) = \int_0^{\delta_1} (f_1 - f_2)' dt > [\mu(\delta_2) - \mu(\delta_1)] \int_0^{\delta_1} (\sin t)^{n-1} dt.$$

Therefore

$$(3.18) \quad \mu(\delta_1) > \mu(\delta_2) + \frac{f_2(\delta_1)}{\int_0^{\delta_1} (\sin t)^{n-1} dt}.$$

Now let  $\delta_1 = \delta \in (0, \frac{\pi}{2})$  and  $\delta_2 = \pi/2$ . Applying the fact that  $\mu(\frac{\pi}{2}) = n$ ,  $J_2 = \sin t$ , and

$$f_2 = (\sin t)^{n-2} \cos t,$$

we have

$$(3.19) \quad \begin{aligned} \mu(\delta) &> n + \frac{(\sin \delta)^{n-2} \cos \delta}{\int_0^{\delta} (\sin t)^{n-1} dt} \\ &> n + \frac{(\sin \delta)^{n-2} \cos^2 \delta}{\int_0^{\delta} \cos t (\sin t)^{n-1} dt} \\ &= \frac{n}{\sin^2 \delta}. \end{aligned}$$

Therefore, (ii) is proved.  $\square$

#### 4. A VOLUME ESTIMATE ON DOMAINS IN $\mathbb{R}^n$

On  $\mathbb{R}^n$ , the standard Euclidean metric  $\bar{g}$  satisfies  $DR_{\bar{g}}^*(\lambda) = \bar{g}$  with

$$(4.1) \quad \lambda(x) = -\frac{1}{2(n-1)}|x - a|^2 + L$$

where  $|\cdot|$  denotes the Euclidean length,  $a \in \mathbb{R}^n$  is any fixed point and  $L$  is an arbitrary constant. In this section, we use this fact and Corollary 2.1 to prove Theorem 1.4 in the introduction. First we need some lemmas.

**Lemma 4.1.** *On a compact Riemannian manifold  $(\Omega, \bar{g})$  with smooth boundary  $\Sigma$ , there exists a positive constant  $C$  depending only on  $(\Omega, \bar{g})$  such that, for any Lipschitz function  $\phi$  on  $\Sigma$ , there is an extension of  $\phi$  to a Lipschitz function  $\tilde{\phi}$  on  $\Omega$  such that*

$$(4.2) \quad \int_{\Omega} \left( |\tilde{\phi}|^2 + |\bar{\nabla} \tilde{\phi}|^2 \right) d\text{vol}_{\bar{g}} \leq C \int_{\Sigma} \left( \phi^2 + |\bar{\nabla}^{\Sigma} \phi|^2 \right) d\sigma_{\bar{g}}$$

where  $\bar{\nabla}$ ,  $\bar{\nabla}^{\Sigma}$  denote the gradient on  $\Omega$ ,  $\Sigma$  respectively.

*Proof.* Let  $d(\cdot, \Sigma)$  be the distance to  $\Sigma$ . Let  $\delta > 0$  be a small constant such that the tubular neighborhood  $U_{2\delta} = \{x \in \Omega \mid d(x, \Sigma) < 2\delta\}$  can be parametrized by  $F : \Sigma \times [0, 2\delta) \rightarrow U_{2\delta}$ , with  $F(y, t) = \exp_y(t\nu(y))$  where  $\exp_y(\cdot)$  is the exponential map at  $y \in \Sigma$  and  $\nu(y)$  is the inward unit normal at  $y$ . In  $U_{2\delta}$ , the metric  $\bar{g}$  takes the form  $dt^2 + \sigma^t$ , where  $\{\sigma^t\}_{0 \leq t < 2\delta}$  is a family of metrics on  $\Sigma$ . By choosing  $\delta$  sufficiently small, one can assume  $\sigma^t$  is equivalent to  $\sigma^0$  in the sense that  $\frac{1}{2} \leq \sigma^t(v, v) \leq 2$  for any tangent vector  $v$  with  $\sigma^0(v, v) = 1$ ,  $\forall 0 \leq t < 2\delta$ .

Let  $\rho = \rho(t)$  be a fixed smooth cut-off function on  $[0, \infty)$  such that  $0 \leq \rho \leq 1$ ,  $\rho(t) = 1$  for  $0 \leq t \leq \delta$  and  $\rho(t) = 0$  for  $t \geq \frac{3}{2}\delta$ . On  $U_{2\delta}$ , consider the function  $\tilde{\phi}(y, t) = \phi(y)\rho(t)$ . Since  $\tilde{\phi}$  is identically zero outside  $U_{\frac{3}{2}\delta} = \{x \in \Omega \mid d(x, \Sigma) < \frac{3}{2}\delta\}$ ,  $\tilde{\phi}$  can be viewed as an extension of  $\phi$  on  $\Omega$ . For such an  $\tilde{\phi}$ , one has

$$(4.3) \quad \int_{\Omega} |\tilde{\phi}|^2 d\text{vol}_{\bar{g}} \leq \int_0^{2\delta} \left( \int_{\Sigma} |\phi|^2 d\sigma^t \right) dt \leq C\delta \int_{\Sigma} |\phi|^2 d\sigma_{\bar{g}}$$

and

$$(4.4) \quad \begin{aligned} \int_{\Omega} |\bar{\nabla} \tilde{\phi}|^2 d\text{vol}_{\bar{g}} &\leq 2 \int_{U_{2\delta}} (|\bar{\nabla} \rho|^2 \phi^2 + |\bar{\nabla} \phi|^2 \rho^2) d\text{vol}_{\bar{g}} \\ &\leq C\delta \int_{\Sigma} |\phi|^2 d\sigma_{\bar{g}} + 2 \int_0^{2\delta} \left( \int_{\Sigma} |\bar{\nabla}_t^{\Sigma} \phi|^2 d\sigma^t \right) dt \\ &\leq C \left[ \int_{\Sigma} |\phi|^2 d\sigma_{\bar{g}} + \int_{\Sigma} |\bar{\nabla}^{\Sigma} \phi|^2 d\sigma_{\bar{g}} \right] \end{aligned}$$

where  $\bar{\nabla}_t^{\Sigma}$  denotes the gradient on  $(\Sigma, \sigma^t)$  and  $C$  is a positive constant depending only on  $(\Omega, \bar{g})$ . (4.2) now follows from (4.3) and (4.4).  $\square$

**Lemma 4.2.** *On a compact Riemannian manifold  $(\Omega, \bar{g})$  with smooth boundary  $\Sigma$ , there exists a positive constant  $C$  depending only on  $(\Omega, \bar{g})$  such that, for any smooth  $(0, 2)$  symmetric tensor  $h$  on  $\Omega$ , one has*

$$(4.5) \quad \int_{\Omega} |h|^3 d\text{vol}_{\bar{g}} \leq C \left( \int_{\Sigma} |h|^3 d\sigma_{\bar{g}} + \|h\|_{C^2(\Omega)} \int_{\Sigma} |h|^2 d\sigma_{\bar{g}} + \int_{\Omega} |h| |\bar{\nabla} h|^2 d\text{vol}_{\bar{g}} \right)$$

*Proof.* On  $\Omega$ , let  $\phi = |h|^{\frac{3}{2}}$ . By lemma 4.1, there exists a Lipschitz function  $\tilde{\phi}$  on  $\Omega$  such that  $\tilde{\phi}|_{\Sigma} = \phi|_{\Sigma}$  and

$$\int_{\Omega} \left( |\tilde{\phi}|^2 + |\bar{\nabla} \tilde{\phi}|^2 \right) d\text{vol}_{\bar{g}} \leq C \int_{\Sigma} \left( \phi^2 + |\bar{\nabla}^{\Sigma} \phi|^2 \right) d\sigma_{\bar{g}}.$$

Let  $\lambda_1 > 0$  be the first Dirichlet eigenvalue of  $(\Omega, \bar{g})$ , then

$$(4.6) \quad \begin{aligned} \int_{\Omega} \phi^2 d\text{vol}_{\bar{g}} &\leq 2 \int_{\Omega} \left[ \tilde{\phi}^2 + (\phi - \tilde{\phi})^2 \right] d\text{vol}_{\bar{g}} \\ &\leq 2 \int_{\Omega} \tilde{\phi}^2 d\text{vol}_{\bar{g}} + 2\lambda_1^{-1} \int_{\Omega} |\bar{\nabla}(\phi - \tilde{\phi})|^2 d\text{vol}_{\bar{g}} \\ &\leq C \left[ \int_{\Sigma} \left( \phi^2 + |\bar{\nabla}^{\Sigma} \phi|^2 \right) d\sigma_{\bar{g}} + \int_{\Omega} |\bar{\nabla} \phi|^2 d\text{vol}_{\bar{g}} \right] \end{aligned}$$

where

$$(4.7) \quad \int_{\Omega} |\bar{\nabla} \phi|^2 d\text{vol}_{\bar{g}} = \int_{\Omega} |\bar{\nabla} |h|^{\frac{3}{2}}|^2 d\text{vol}_{\bar{g}} \leq \frac{9}{4} \int_{\Omega} |h| |\bar{\nabla} h|^2 d\text{vol}_{\bar{g}}.$$

To handle the boundary term  $\int_{\Sigma} |\bar{\nabla}^{\Sigma} \phi|^2 d\sigma_{\bar{g}}$ , given any constant  $\epsilon > 0$ , one considers

$$(4.8) \quad \int_{\Sigma} |\bar{\nabla}^{\Sigma} (|h|^2 + \epsilon)^{\frac{3}{4}}|^2 d\sigma_{\bar{g}} = - \int_{\Sigma} (|h|^2 + \epsilon)^{\frac{3}{4}} \Delta_{\Sigma} (|h|^2 + \epsilon)^{\frac{3}{4}} d\sigma_{\bar{g}}$$

where  $\Delta_{\Sigma}$  denotes the Laplacian on  $\Sigma$ . Let  $\{e_{\alpha} \mid \alpha = 1, \dots, n-1\}$  be a local orthonormal frame on  $\Sigma$  and  $e_n$  be the outward unit normal to  $\Sigma$ . Let  $\bar{H}$  be the mean curvature of  $\Sigma$  with respect to  $e_n$ . Denote covariant differentiation on  $\Omega$  by “;”. Let  $i, j$  run through  $\{1, \dots, n\}$ . One has

$$(4.9) \quad \begin{aligned} \Delta_{\Sigma} |h|^2 &= \sum_{\alpha} (|h|^2)_{;\alpha\alpha} - \bar{H} (|h|^2)_{;n} \\ &= \sum_{\alpha, i, j} 2(h_{ij} h_{ij;\alpha\alpha} + h_{ij;\alpha}^2) - \bar{H} \sum_{i, j} 2h_{ij} h_{ij;n} \\ &\geq -C \|h\|_{C^2(\bar{\Omega})} |h|. \end{aligned}$$

Therefore,

(4.10)

$$\begin{aligned} \Delta_\Sigma(|h|^2 + \epsilon)^{\frac{3}{4}} &= \frac{3}{4}(|h|^2 + \epsilon)^{-\frac{1}{4}} \Delta_\Sigma |h|^2 - \frac{3}{16}(|h|^2 + \epsilon)^{-\frac{5}{4}} |\bar{\nabla}^\Sigma |h|^2|^2 \\ &\geq -C \|h\|_{C^2(\bar{\Omega})} (|h|^2 + \epsilon)^{-\frac{1}{4}} |h| - \frac{3}{16}(|h|^2 + \epsilon)^{-\frac{5}{4}} |\bar{\nabla}^\Sigma |h|^2|^2. \end{aligned}$$

It follows from (4.8) and (4.10) that

$$\begin{aligned} \int_\Sigma |\bar{\nabla}^\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}}|^2 d\sigma_{\bar{g}} &\leq C \|h\|_{C^2(\bar{\Omega})} \int_\Sigma (|h|^2 + \epsilon)^{\frac{1}{2}} |h| d\sigma_{\bar{g}} \\ &\quad + \frac{1}{3} \int_\Sigma |\bar{\nabla}^\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}}|^2 d\sigma_{\bar{g}}. \end{aligned} \quad (4.11)$$

Letting  $\epsilon \rightarrow 0$ , one has

$$\int_\Sigma |\bar{\nabla}^\Sigma |h|^{\frac{3}{2}}|^2 d\sigma_{\bar{g}} \leq C \|h\|_{C^2(\bar{\Omega})} \int_\Sigma |h|^2 d\sigma_{\bar{g}}. \quad (4.12)$$

(4.5) now follows from (4.6), (4.7) and (4.12).  $\square$

We recall the statement of Theorem 1.4 and give its proof.

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\Sigma$ . Suppose  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} > 0$  (i.e.  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma}$  is positive definite), where  $\bar{\mathbb{I}}\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$  in  $\mathbb{R}^n$  and  $\bar{\gamma}$  is the metric on  $\Sigma$  induced from the Euclidean metric  $\bar{g}$ . Let  $g$  be another metric on  $\bar{\Omega}$  satisfying*

- $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$ .
- $H(g) \geq \bar{H}$ , where  $H(g)$  is the mean curvature of  $\Sigma$  in  $(\Omega, g)$ .

*Given any point  $a \in \mathbb{R}^n$ , there exists a constant  $\Lambda > \frac{\max_{q \in \bar{\Omega}} |q-a|^2}{4(n-1)}$ , which depends only on  $\Omega$  and  $a$ , such that if  $\|g - \bar{g}\|_{C^3(\bar{\Omega})}$  is sufficiently small, then*

$$V(g) - V(\bar{g}) \geq \int_\Omega R(g) \Phi \, d\text{vol}_{\bar{g}} \quad (4.13)$$

where  $\Phi = -\frac{1}{4(n-1)}|x - a|^2 + \Lambda > 0$  on  $\bar{\Omega}$ .

*Proof.* Fix a number  $p > n$ . By the proof of [5, Proposition 11], one knows if  $\|g - \bar{g}\|_{W^{3,p}(\Omega)}$  is sufficiently small, then there exists a  $W^{4,p}$  diffeomorphism  $\varphi : \Omega \rightarrow \Omega$  such that  $\varphi|_\Sigma = \text{id}$ ,  $h = \varphi^*(g) - \bar{g}$  is divergence free with respect to  $\bar{g}$ , and  $\|h\|_{W^{3,p}(\Omega)} \leq N\|g - \bar{g}\|_{W^{3,p}(\Omega)}$  for a positive constant  $N$  depending only on  $(\Omega, \bar{g})$ . In what follows, we will work with  $\phi^*(g)$ . For convenience, we still denote  $\phi^*(g)$  by  $g$ .

Given  $a \in \mathbb{R}^n$ , consider  $\lambda(x) = -\frac{1}{2(n-1)}|x - a|^2 + L$  where  $L$  is a constant to be determined. First, we require  $L > \frac{1}{2(n-1)} \max_{q \in \bar{\Omega}} |q - a|^2$  so that  $\lambda > 0$  on  $\bar{\Omega}$ . Since  $\lambda$  satisfies  $DR_{\bar{g}}^*(\lambda) = \bar{g}$ , Corollary 2.1 shows

$$\begin{aligned}
 (4.14) \quad & -2(V(g) - V(\bar{g})) + \int_{\Omega} R(g)\lambda \, d\text{vol}_{\bar{g}} + \int_{\Sigma} (2 - \text{tr}_{\bar{g}}h) [H(g) - \bar{H}] \lambda \, d\sigma_{\bar{g}} \\
 & \leq - \int_{\Omega} \frac{1}{4} |\bar{\nabla}h|^2 \lambda \, d\text{vol}_{\bar{g}} + \int_{\Sigma} \left[ -\frac{1}{4} (h_{nn})^2 \bar{H} - \frac{1}{2} (\bar{\mathbb{I}}\mathbb{I}(X, X) + \bar{H}|X|^2) \right] \lambda \, d\sigma_{\bar{g}} \\
 & \quad + \int_{\Sigma} \lambda_{;n} \left[ -(h_{nn})^2 - \frac{1}{2} |X|^2 \right] d\sigma_{\bar{g}} + \int_{\Sigma} (-1) h_{nn} \langle X, \bar{\nabla}^{\Sigma} \lambda \rangle d\sigma_{\bar{g}} \\
 & \quad + \int_{\Omega} G(h) \, d\text{vol}_{\bar{g}} + \int_{\Omega} E(h) \lambda \, d\text{vol}_{\bar{g}} + \int_{\Omega} Z^i(h) \bar{\nabla}_i \lambda \, d\text{vol}_{\bar{g}} + \int_{\Sigma} F(h) \lambda \, d\sigma_{\bar{g}}
 \end{aligned}$$

where  $|G(h)| \leq C|h|^3$ ,  $|E(h)| \leq C(|h||\bar{\nabla}h|^2 + |h|^3)$ ,  $|Z(h)| \leq C|h|^2|\bar{\nabla}h|$ ,  $|F(h)| \leq C(|h|^2|\bar{\nabla}h| + |h|^3)$  for some constant  $C$  depending only on  $\Omega$ .

At  $\Sigma$ ,  $\lambda_{;n}$  and  $\bar{\nabla}^{\Sigma} \lambda$  are determined solely by  $\Omega$  and  $a$  (in particular they are independent on  $L$ ). Apply the assumption  $\bar{\mathbb{I}}\mathbb{I} + \bar{H}\bar{\gamma} > 0$  (which implies  $\bar{H} > 0$ ) and the fact  $|h|^2 = (h_{nn})^2 + 2|X|^2$ , we have

$$\begin{aligned}
 (4.15) \quad & \left[ -\frac{1}{4} (h_{nn})^2 \bar{H} - \frac{1}{2} (\bar{\mathbb{I}}\mathbb{I}(X, X) + \bar{H}|X|^2) \right] \lambda \\
 & + \lambda_{;n} \left[ -(h_{nn})^2 - \frac{1}{2} |X|^2 \right] + (-1) h_{nn} \langle X, \bar{\nabla}^{\Sigma} \lambda \rangle \\
 & \leq -LC_1|h|^2 + C_2|h|^2
 \end{aligned}$$

where  $C_1, C_2$  are positive constants depending only on  $\Omega$  and  $a$ . We fix  $L$  such that

$$(4.16) \quad LC_1 - C_2 > 0$$

and let  $m = \frac{1}{4} \min_{\bar{\Omega}} \lambda$  (note that  $\lambda$  is fixed now). (4.14)-(4.16) imply

$$\begin{aligned}
 (4.17) \quad & -2(V(g) - V(\bar{g})) + \int_{\Omega} R(g)\lambda \, d\text{vol}_{\bar{g}} + \int_{\Sigma} (2 - \text{tr}_{\bar{g}}h) [H(g) - \bar{H}] \lambda \, d\sigma_{\bar{g}} \\
 & \leq -m \int_{\Omega} |\bar{\nabla}h|^2 \, d\text{vol}_{\bar{g}} - (LC_1 - C_2) \int_{\Sigma} |h|^2 d\sigma_{\bar{g}} \\
 & \quad + C_3 \left( \int_{\Omega} (|h||\bar{\nabla}h|^2 + |h|^3) d\text{vol}_{\bar{g}} + \int_{\Sigma} (|h|^2|\bar{\nabla}h| + |h|^3) d\sigma_{\bar{g}} \right)
 \end{aligned}$$

where  $C_3$  depends only on  $\Omega$ ,  $a$  and  $L$ . Apply Lemma 4.2 to the term  $\int_{\Omega} |h|^3 d\text{vol}_{\bar{g}}$  on the right side of (4.17), we have

$$\begin{aligned} & -2(V(g) - V(\bar{g})) + \int_{\Omega} R(g)\lambda d\text{vol}_{\bar{g}} + \int_{\Sigma} (2 - \text{tr}_{\bar{g}}h) [H(g) - \bar{H}] \lambda d\sigma_{\bar{g}} \\ \leq & -m \int_{\Omega} |\bar{\nabla}h|^2 d\text{vol}_{\bar{g}} - (LC_1 - C_2) \int_{\Sigma} |h|^2 d\sigma_{\bar{g}} \\ & + C \|h\|_{C^2(\bar{\Omega})} \left( \int_{\Omega} |\bar{\nabla}h|^2 d\text{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 d\sigma_{\bar{g}} \right). \end{aligned}$$

where  $C$  is independent on  $h$ . From this, we conclude that if  $\|h\|_{C^2(\bar{\Omega})}$  is sufficiently small, then (4.13) holds with  $\Phi = \frac{1}{2}\lambda$ . This completes the proof.  $\square$

*Remark 4.1.* When  $\Omega \subset \mathbb{R}^n$  is a ball of radius  $R$ , one can take  $a$  to be the center of  $\Omega$ . In this case, by computing  $\bar{H}$ ,  $\bar{\text{III}}$  and  $\lambda_n$  explicitly in (4.16), the constant  $L$  can be chosen to be any constant satisfying

$$L > \left[ \frac{1}{2(n-1)} + \frac{4}{(n-1)^2} \right] R^2.$$

*Remark 4.2.* By the results in [12, 17] based on the positive mass theorem [16, 18], a metric  $g$  on  $\Omega$  satisfying the boundary conditions in Theorem 4.1 must be isometric to the Euclidean metric if  $R(g) \geq 0$ . Therefore, a nontrivial metric  $g$  in Theorem 4.1 necessarily has negative scalar curvature somewhere. For such a  $g$ , Theorem 4.1 shows if the weighted integral  $\int_{\Omega} R(g)\Phi d\text{vol}_{\bar{g}}$  is nonnegative, then  $V(g) \geq V(\bar{g})$ .

## 5. OTHER RELATED RESULTS

In this section, we collect some other by-products of the formulas derived in Section 2. First, we discuss a scalar curvature rigidity result for general domains in  $\mathbb{S}^n$ .

**Theorem 5.1.** *Let  $\Omega \subset \mathbb{S}^n$  be a smooth domain contained in a geodesic ball  $B$  of radius less than  $\frac{\pi}{2}$ . Let  $\bar{g}$  be the standard metric on  $\mathbb{S}^n$ . Let  $\bar{\text{III}}$ ,  $\bar{H}$  be the second fundamental form, the mean curvature of  $\Sigma = \partial\Omega$  in  $(\Omega, \bar{g})$  with respect to the outward unit normal  $\bar{\nu}$ . Suppose  $\bar{\text{III}} \geq -c\bar{\gamma}$ , where  $c \geq 0$  is a function on  $\Sigma$  and  $\bar{\gamma}$  is the induced metric on  $\Sigma$ . Let  $q$  be the center of  $B$ . Suppose at  $\Sigma \setminus \{q\}$ ,*

$$(5.1) \quad \bar{H} - c \geq \left[ \frac{5 \cos \theta + \sqrt{\cos^2 \theta + 8}}{2} \right] \tan r$$

where  $r$  is the  $\bar{g}$ -distance to  $q$  and  $\theta$  is the angle between  $\bar{\nu}$  and  $\bar{\nabla}r$ . Then the conclusion of Theorem 1.6 holds on  $\Omega$ .

*Proof.* As before, replacing  $g$  by  $\varphi^*(g)$  for some diffeomorphism  $\varphi$ , we may assume  $\operatorname{div}_{\bar{g}} h = 0$  where  $h = g - \bar{g}$ . On  $\Omega$ , let  $\lambda = \cos r > 0$ , where  $r$  is the  $\bar{g}$ -distance to  $q$ . At  $\Sigma \setminus \{q\}$ , we have

$$(5.2) \quad \lambda_{;n} = -\sin r \cos \theta, \quad |\bar{\nabla}^\Sigma \lambda| = \sin r \sin \theta.$$

Apply Theorem 2.1, using the fact  $DR_{\bar{g}}^*(\lambda) = 0$  and the assumptions on  $R(g)$  and  $H(g)$ , we have

$$(5.3) \quad \begin{aligned} & \int_{\Omega} \left[ \frac{1}{4} (|\bar{\nabla} h|^2 + |\bar{\nabla}(\operatorname{tr}_{\bar{g}} h)|^2) + \frac{1}{2} (|h|^2 + (\operatorname{tr}_{\bar{g}} h)^2) \right] \cos r \, d\operatorname{vol}_{\bar{g}} \\ & \leq \int_{\Sigma} \left[ -\frac{1}{4} (h_{nn})^2 \bar{H} - \frac{1}{2} (\bar{\nabla}^{\Sigma}(X, X) + \bar{H}|X|^2) \right] \cos r \, d\sigma_{\bar{g}} \\ & \quad + \int_{\Sigma \setminus \{q\}} \left[ (h_{nn})^2 + \frac{1}{2} |X|^2 \right] (\sin r \cos \theta) \, d\sigma_{\bar{g}} + \int_{\Sigma \setminus \{q\}} |h_{nn}| |X| (\sin r \sin \theta) \, d\sigma_{\bar{g}} \\ & \quad + C \|h\|_{C^1(\bar{\Omega})} \left\{ \int_{\Omega} (|h|^2 + |\bar{\nabla} h|^2) \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 \, d\sigma_{\bar{g}} \right\} \\ & \leq - \int_{\Sigma \setminus \{q\}} \left[ \left( \frac{1}{4} (\bar{H} - c) \cos r - \sin r \cos \theta \right) (h_{nn})^2 + \frac{1}{2} ((\bar{H} - c) \cos r - \sin r \cos \theta) |X|^2 \right. \\ & \quad \left. - |h_{nn}| |X| (\sin r \sin \theta) \right] d\sigma_{\bar{g}} \\ & \quad + C \|h\|_{C^1(\bar{\Omega})} \left\{ \int_{\Omega} (|h|^2 + |\bar{\nabla} h|^2) \, d\operatorname{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 \, d\sigma_{\bar{g}} \right\} \end{aligned}$$

for some positive constant  $C$  independent on  $h$ .

Note that the assumption (5.1) implies

$$(5.4) \quad \frac{1}{4} (\bar{H} - c) \cos r - (\sin r \cos \theta) \geq 0$$

and

$$(5.5) \quad (\bar{H} - c) \cos r - (\sin r \cos \theta) \geq 0.$$

By (5.1), (5.4) and (5.5), we have

$$(5.6) \quad \begin{aligned} 0 & \leq \left( \frac{1}{4} (\bar{H} - c) \cos r - \sin r \cos \theta \right) (h_{nn})^2 - |h_{nn}| |X| (\sin r \sin \theta) \\ & \quad + \frac{1}{2} ((\bar{H} - c) \cos r - \sin r \cos \theta) |X|^2 \end{aligned}$$

for any  $h_{nn}$  and  $X$ . The result now follows from (5.3) and (5.6).  $\square$

*Remark 5.1.* It is clear from the proof of Theorem 5.1 that the center  $q$  of  $B$  does not need to be inside  $\Omega$ .

Theorem 5.1 directly implies Theorem 1.7 in the introduction.

*Proof of Theorem 1.7.* Choose  $c = 0$  in Theorem 5.1. Since

$$4 \geq \frac{5 \cos \theta + \sqrt{\cos^2 \theta + 8}}{2}$$

for any  $\theta$ , the result follows from Theorem 5.1.  $\square$

Next, we consider a corresponding scalar curvature rigidity result when the background metric  $\bar{g}$  is a flat metric.

**Theorem 5.2.** *Let  $\Omega$  be a compact manifold with smooth boundary  $\Sigma$ . Suppose  $\bar{g}$  is a smooth Riemannian metric on  $\Omega$  such that  $\bar{g}$  has zero sectional curvature and  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$  on  $\Sigma$ , where  $\bar{\mathbb{I}}\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$ , and  $\bar{\gamma}$  is the induced metric on  $\Sigma$ . Suppose  $g$  is another metric on  $\Omega$  satisfying*

- $R(g) \geq 0$  where  $R(g)$  is the scalar curvature of  $g$
- $g$  and  $\bar{g}$  induce the same metric on  $\Sigma$
- $H(g) \geq \bar{H}$  where  $H(g)$  is the mean curvature of  $\Sigma$  in  $(\Omega, g)$ .

*If  $\|g - \bar{g}\|_{C^2(\bar{\Omega})}$  is sufficiently small, then there is a diffeomorphism  $\varphi$  on  $\Omega$  with  $\varphi|_{\Sigma} = \text{id}$  such that  $\varphi^*(g) = \bar{g}$ .*

*Proof.* As before, we may assume  $\text{div}_{\bar{g}} h = 0$  where  $h = g - \bar{g}$ . Choose  $\lambda = 1$  in (2.23), one has

$$\begin{aligned} & \int_{\Omega} \left[ \frac{1}{4} (|\bar{\nabla} h|^2 + |\bar{\nabla}(\text{tr}_{\bar{g}} h)|^2) \right] d\text{vol}_{\bar{g}} \\ (5.7) \quad & + \int_{\Sigma} \left[ \frac{1}{4} (h_{nn})^2 H(\bar{g}) + \frac{1}{2} (\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) + H(\bar{g})|X|^2) \right] d\sigma_{\bar{g}} \\ & \leq \int_{\Omega} E(h) d\text{vol}_{\bar{g}} + \int_{\Sigma} F(h) d\sigma_{\bar{g}} \end{aligned}$$

where  $|F(h)| \leq C(|h|^2 |\bar{\nabla} h| + |h|^3)$  and  $|E(h)| \leq C|h| |\bar{\nabla} h|^2$  by Remark 2.1. The result follows from (5.7).  $\square$

To finish, we mention that the positive Gaussian curvature condition of the boundary surface in [17] is not a necessary condition for the positivity of its Brown-York mass.

**Theorem 5.3.** *Let  $\Sigma \subset \mathbb{R}^n$  be a connected, closed hypersurface satisfying  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$ , where  $\bar{\mathbb{I}}\bar{\mathbb{I}}$ ,  $\bar{H}$  are the second fundamental form, the mean curvature of  $\Sigma$ , and  $\bar{\gamma}$  is the induced metric on  $\Sigma$ . Let  $\Omega$  be the domain enclosed by  $\Sigma$  in  $\mathbb{R}^n$ . Let  $h$  be any nontrivial  $(0, 2)$  symmetric tensor on  $\Omega$  satisfying*

$$(5.8) \quad \text{div}_{\bar{g}} h = 0, \quad \text{tr}_{\bar{g}} h = 0, \quad h|_{T\Sigma} = 0.$$



Let  $\{g(t)\}_{|t|<\epsilon}$  be a 1-parameter family of metrics on  $\Omega$  satisfying

$$(5.9) \quad g(0) = \bar{g}, \quad g'(0) = h, \quad R(g(t)) \geq 0, \quad g(t)|_{T\Sigma} = \bar{g}|_{T\Sigma}.$$

Then

$$(5.10) \quad \int_{\Sigma} \bar{H} d\sigma_{\bar{g}} > \int_{\Sigma} H(g(t)) d\sigma_{\bar{g}}$$

for small  $t \neq 0$ , where  $H(g(t))$  is the mean curvature of  $\Sigma$  in  $(\Omega, g(t))$ .

*Proof.* By Lemma 2.2, one knows

$$\frac{d}{dt} \left( \int_{\Omega} [R(g(t)) - R(\bar{g})] d\text{vol}_{\bar{g}} - 2 \int_{\Sigma} [\bar{H} - H(g(t))] d\sigma_{\bar{g}} \right) \Big|_{t=0} = 0.$$

Direct calculation using Lemma 2.2, (2.17) and (5.8) shows

$$(5.11) \quad \begin{aligned} & \frac{d^2}{dt^2} \left( \int_{\Omega} [R(g(t)) - R(\bar{g})] d\text{vol}_{\bar{g}} - 2 \int_{\Sigma} [\bar{H} - H(g(t))] d\sigma_{\bar{g}} \right) \Big|_{t=0} \\ &= -\frac{1}{2} \int_{\Omega} |\bar{\nabla} h|^2 d\text{vol}_{\bar{g}} - \int_{\Sigma} [(\bar{\mathbb{I}}\bar{\mathbb{I}}(X, X) + H(\bar{g})|X|^2)] d\sigma_{\bar{g}} \end{aligned}$$

which is negative by the assumption on  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma}$ . Thus, for small  $t$ ,

$$(5.12) \quad 2 \int_{\Sigma} [\bar{H} - H(g(t))] d\sigma_{\bar{g}} > \int_{\Omega} [R(g(t)) - R(\bar{g})] d\text{vol}_{\bar{g}} \geq 0.$$

□

Given an  $h$  satisfying (1.6), a family of deformation  $\{g(t)\}$  satisfying (1.7) is given by  $g(t) = u(t)^{\frac{4}{n-2}}(\bar{g} + th)$  for small  $t$ , where  $u(t) > 0$  is a conformal factor such that  $R(g(t)) = 0$  (see [13, Lemma 4]).

An example of a non-convex surface  $\Sigma \subset \mathbb{R}^3$ , which is topologically a 2-sphere and satisfies the condition  $\bar{\mathbb{I}}\bar{\mathbb{I}} + \bar{H}\bar{\gamma} \geq 0$ , is given by a capsule-shaped surface with its middle slightly pinched.

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